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On the Zeros of Sequences of Polynomials

R. GROTHMANN

Katholische Universität, Eichstatt 8078, West Germany

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In this paper we generalize a result of Blatt, Saff, and Simkani on the limit distribution of zeros of sequences of polynomials. In a typical application these polynomials converge on a compact subset E of the complex plane. The highest coefficient of the polynomials plays an important role in the theorem of Blatt, Saff, and Simkani. In this paper we replace the behavior of the highest coefficient by the behavior of the sequence on some compact set in $\bar{\mathbb{C}}\setminus E$. Furthermore we show how this generalization can be applied to sequences of maximally convergent polynomials.

1. INTRODUCTION AND STATEMENT OF RESULTS

Throughout this paper, E will be a compact subset of $\mathbb{C}$, the complex plane, such that $\mathbb{C}\setminus E$ is connected and regular, i.e., there is a Green's function G on $\mathbb{C}\setminus E$ with pole at ∞:

$$G(z) \text{ is harmonic in } \mathbb{C}\setminus E, \tag{1.1}$$

$$G(z) - \log |z| \text{ is harmonic at } \infty, \tag{1.2}$$

$$\lim_{z \to \partial E} G(z) = 0. \tag{1.3}$$

Then

$$\lim_{z \to \infty} (G(z) - \log |z|) = -\log \operatorname{cap} E, \tag{1.4}$$

where $\operatorname{cap} E$ is the capacity of E (compare [6]). If we denote by μ_E the equilibrium measure of E and by U^{μ_E} its logarithmic potential, then

$$\begin{aligned} G(z) &= -U^{\mu_E}(z) - \log \operatorname{cap} E \\ &= -\left[\int \log \frac{1}{|z-\zeta|}\, d\mu_E(\zeta)\right] - \log \operatorname{cap} E \end{aligned} \tag{1.5}$$

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(compare [6]). We set $\bar{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. Furthermore, we will assume that (p_n) is a sequence of complex polynomials, such that $p_n \in \Pi_{k(n)}(\mathbb{C})$ and $k(n)$ is the exact degree of p_n. Defining the zero-measure ν_n associated with p_n as

$$\nu_n(A) := \frac{\#\text{ of zeros of } p_n \text{ in } A}{\text{degree } p_n} \quad \text{for} \quad A \subseteq \mathbb{C}, \tag{1.6}$$

where the zeros are counted with their multiplicity, we establish the following theorem.

THEOREM 1. *Assume the following conditions hold:*

(1)

$$\limsup_{n \to \infty} \left(\sup_{z \in E} \frac{1}{k(n)} \log |p_n(z)| \right) \leqslant 0. \tag{1.7}$$

(2) *For every compact $M \subseteq E^\circ$, where E° denotes the open interior of E,*

$$\lim_{n \to \infty} \nu_n(M) = 0. \tag{1.8}$$

(3) *There is a compact set $K \subseteq \bar{\mathbb{C}} \setminus E$ with*

$$\liminf_{n \to \infty} \left[\sup_{z \in K} \left(\frac{1}{k(n)} \log |p_n(z)| - G(z) \right) \right] \geqslant 0. \tag{1.9}$$

Then

$$\nu_n \to \mu_E \tag{1.10}$$

weakly for $n \to \infty$.

Weak convergence is equivalent to

$$\lim_{n \to \infty} \int \phi \, d\nu_n = \int \phi \, d\mu_E \tag{1.11}$$

for all continuous ϕ with compact support.

The proof of Theorem 1 is contained in Section 2. Of course, (1) and (2) are satisfied, if the sequence (p_n) converges uniformly on E to a function, which does not vanish on any connected open component of E°. If (3) holds, we call $G(z)$ an exact harmonic majorant of $1/k(n) \log |p_n(z)|$ on K (compare [8, 9]).

Remark 1. If we define

$$h_n(z) := \frac{1}{k(n)} \log |p_n(z)| - G(z), \tag{1.12}$$

then $h_n(z)$ is subharmonic in $\bar{\mathbb{C}} \setminus E$ and harmonic in ∞. By the maximum principle for subharmonic functions and (1.7), we have equality in (1.7) and (1.9). Furthermore, "lim inf" and "lim sup" can both be replaced by "lim." We show now, that $\lim_{n \to \infty} k(n) = \infty$. For otherwise, the polynomials have a limit point p, which has no zeros in E° by (1.8) and no zeros in $\bar{\mathbb{C}} \setminus E^\circ$ by (1.9), a contradiction.

Remark 2. Theorem 1 remains true, if we replace $k(n)$ = degree of p_n by some sequence $\tilde{k}(n) \geqslant k(n)$ throughout Theorem 1. To see this, assume (1.7) and (1.9) hold with $k(n)$ replaced by $\tilde{k}(n)$. Then, by the maximum principle for subharmonic functions,

$$\begin{aligned} &\frac{1}{\tilde{k}(n)} \log |p_n(z)| - G(z) \\ &\quad = \frac{1}{\tilde{k}(n)} (\log |p_n(z)| - k(n)\, G(z)) + \left(\frac{k(n)}{\tilde{k}(n)} - 1\right) G(z) \\ &\quad \leqslant \frac{1}{\tilde{k}(n)} \sup_{z \in \partial E} \log |p_n(z)| + \left(\frac{k(n)}{\tilde{k}(n)} - 1\right) G(z) \end{aligned} \tag{1.13}$$

for all $z \in K$. Thus, by (1.7) and (1.9),

$$0 \leqslant \liminf_{n \to \infty} \left(\frac{k(n)}{\tilde{k}(n)} - 1\right) \inf_{z \in K} G(z), \tag{1.14}$$

from which $\lim_{n \to \infty} k(n)/\tilde{k}(n) = 1$ follows. Therefore, (1.7) and (1.9) hold for the sequence $k(n)$ too.

With $K = \{\infty\}$, we get the theorem of Blatt, Saff, and Simkani [1] for regular sets E.

COROLLARY 1 (Blatt, Saff, Simkani). *Let $q_n = z^n + \cdots$ be a sequence of monic polynomials of degree n, such that*

$$\liminf_{n \to \infty} \|q_n\|_E^{1/n} \leqslant \operatorname{cap} E, \tag{1.15}$$

where $\|\cdot\|$ denotes the sup-norm on E. Assume that q_n has only $o(n)$ zeros in every compact set $M \subseteq E^\circ$ for $n \to \infty$. Then the zero-measures ν_n associated to the q_n have μ_E as a weak limit point.

Proof. Define

$$p_n(z) := \frac{1}{(\text{cap } E)^n} q_n(z). \tag{1.16}$$

Then

$$\liminf_{n \to \infty} h_n(\infty) = 0, \tag{1.17}$$

Thus, for a subsequence, (1.7) and (1.9) hold with $K = \{\infty\}$. Equation (1.8) holds by assumption. The result follows from Theorem 1. ∎

Corollary 1 can be applied, whenever the highest coefficient of the sequence (p_n) is known. This includes the case where the sequence of polynomials is the truncated power series of a function. Then E is the circle of convergence. Truncated power series have been investigated by Jentzsch [4] and Szegö [5]. Szegö obtained the distribution result in Corollary 1. Further quantitative investigations have been done by Erdős and Turán [2].

A special instance of Corollary 1 is the sequence of best uniform approximants on E to a function f, which is continuous on E, analytic in E° and does not vanish on any connected open component of E° (see [1]).

The advantage of Theorem 1 lies in the fact that it does not mention the highest coefficient. An application is a sequence of maximally convergent polynomials. The following definition was used by Walsh [8].

DEFINITION 1. Define, for $\gamma \geqslant 1$,

$$E_\gamma := \{z \in \mathbb{C} \setminus E : G(z) \leqslant \log \gamma\} \cup E, \tag{1.18}$$

and assume that, for some $\gamma > 1$ the function $f: E \to \mathbb{C}$ has an analytic extension on $E_{\tilde{\gamma}}$, for all $\tilde{\gamma} < \gamma$, but not on E_γ. (By an extension on a compact set we mean an extension on a open neighborhood of that set.) Then a sequence of polynomials $p_n \in \Pi_n$ is called maximally convergent to f, if

$$\limsup_{n \to \infty} \|f - p_n\|_E^{1/n} \leqslant \frac{1}{\gamma}. \tag{1.19}$$

Note that we cannot have "<" in (1.19), since otherwise by the generalized Bernstein inequality,

$$\|p_n\|_{E_{\tilde{\gamma}}} \leqslant \tilde{\gamma}^n \|p_n\|_E, \tag{1.20}$$

p_n converges on some $E_{\tilde{\gamma}}$, $\tilde{\gamma} > \gamma$ (for this argument see [8]).

COROLLARY 2. *Assume f has an analytic extension on $E_{\tilde{\gamma}}$ for all $\tilde{\gamma} < \gamma$ but not on E_γ, $\gamma > 1$. Let p_n converge maximally to f, as defined in Definiton 1. Assume that f does not vanish on any closed connected component of E. Then the zero-measures ν_n associated to the p_n have μ_{E_γ} (the equilibrium measure of E_γ) as a weak limit point.*

Proof. We apply Theorem 1 with E replaced by $\tilde{E} := E_\gamma$. By (1.19) and the generalized Bernstein inequality, we have

$$\limsup_{n \to \infty} \|p_n - p_{n+1}\|_{E_{\tilde{\gamma}}}^{1/n} \leqslant \frac{\tilde{\gamma}}{\gamma} \tag{1.21}$$

for all $1 \leqslant \tilde{\gamma} < \gamma$. Thus the sequence (p_n) converges on $E_{\tilde{\gamma}}$ for all $\tilde{\gamma} < \gamma$. Applying (1.20) once more we get

$$\limsup_{n \to \infty} \|p_n\|_{\tilde{E}}^{1/n} \leqslant 1. \tag{1.22}$$

Thus we have (1.7). Since the sequence (p_n) converges on any compact set $M \subseteq \tilde{E}^\circ$, we get (1.8). Let K be any connected compact set $K \subseteq \bar{\mathbb{C}} \setminus \tilde{E}$, such that ∂K is regular, and assume

$$\limsup_{n \to \infty} \left[\sup_{z \in K} \left(\frac{1}{n} \log |p_n(z)| - \tilde{G}(z) \right) \right] = \delta < 0, \tag{1.23}$$

where $\tilde{G}(z) = G(z) - \log \gamma$ denotes the Green's function of $\mathbb{C} \setminus \tilde{E}$. Define a harmonic function g on $\bar{\mathbb{C}} \setminus (E \cup K)$, which tends to 0 on ∂E and to δ on ∂K. By the maximum principle for subharmonic functions and (1.21),

$$\limsup_{n \to \infty} \frac{1}{n} \log |p_n(z) - p_{n+1}(z)| - G(z) \leqslant g(z) - \log \gamma \tag{1.24}$$

for all $z \in \bar{\mathbb{C}} \setminus (E \cup K)$ and $n \in \mathbb{N}$. Thus

$$\limsup_{n \to \infty} \|p_n - p_{n+1}\|_{\tilde{E}}^{1/n} < 1. \tag{1.25}$$

This implies

$$\limsup_{n \to \infty} \|p_n - p_{n+1}\|_{E_{\tilde{\gamma}}}^{1/n} < 1, \tag{1.26}$$

for some $\tilde{\gamma} > \gamma$, using (1.20) for $\tilde{E}$. It follows that the sequence (p_n) converges in $E_{\tilde{\gamma}}$, which is impossible. Thus we have (1.9). Corollary 2 now follows from Theorem 1 and Remark 2. ∎

We remark here that Walsh investigated the zeros of maximally convergent sequences of polynomials [9]. He proved that every point on the boundary of E_γ is a limit point of such zeros, which is a consequence of Corollary 2.

We have very little information on the distribution of the zeros, if (p_n) is a sequence of "near best approximations" to a function f, which is not analytic on E (see [3]). However, Simkani [7] proved a result on sequences of interpolants in the roots of unity, where E is the unit circle.

2. Proof of Theorem 1

Let $\tilde{K}$ be a compact set in $\bar{\mathbb{C}}$, such that $\partial\tilde{K}$ is regular, $\tilde{K} \subseteq \bar{\mathbb{C}} \backslash (E \cup K)$ and $\bar{\mathbb{C}} \backslash (E \cup \tilde{K})$ is connected. Then (1.9) holds with K replaced by $\tilde{K}$. For otherwise, there is a harmonic function ϕ on $\bar{\mathbb{C}} \backslash (E \cup \tilde{K})$, such that

$$\begin{aligned} \phi(t) &= 0 && \text{for} \quad t \in \partial E \\ \phi(t) &= \liminf_{n \to \infty} [\sup_{z \in \tilde{K}} h_n(z)] < 0 && \text{for} \quad t \in \partial\tilde{K}. \end{aligned} \tag{2.1}$$

Thus, by the maximum principle for subharmonic functions,

$$\liminf_{n \to \infty} [\sup_{z \in K} h_n(z)] \leqslant \liminf_{n \to \infty} [\sup_{z \in K} \phi(z)] < 0, \tag{2.2}$$

which contradicts (1.9).

Now we show that in any compact set $S \subseteq \bar{\mathbb{C}} \backslash E$ there are only $o(k(n))$ zeros of p_n. We may assume $K \cap S = \varnothing$. With (1.12) let us define, for $n \in \mathbb{N}$,

$$\tilde{h}_n(z) = h_n(z) + \frac{1}{k(n)} \sum_{\nu} G(z, z_{\nu,n}), \tag{2.3}$$

where $z_{\nu,n}$ are the zeros of p_n in S, and $G(z, w)$ denotes the Green's function on $\bar{\mathbb{C}} \backslash E$ with pole in w. Then $\tilde{h}_n$ is subharmonic in $\bar{\mathbb{C}} \backslash E$. Thus, by the maximum principle for subharmonic functions,

$$\limsup_{n \to \infty} (\sup_{z \in K} \tilde{h}_n(z)) \leqslant 0, \tag{2.4}$$

where we used (1.7). Since $\tilde{h}_n \geqslant h_n$ on $\bar{\mathbb{C}} \backslash E$, we get from (1.9)

$$\lim_{n \to \infty} (\inf_{z \in K} (\tilde{h}_n - h_n)(z)) = 0. \tag{2.5}$$

Let

$$\inf_{z \in K} \inf_{w \in S} G(z, w) = \delta > 0. \tag{2.6}$$

(We have $\delta > 0$, since $G(z, w)$ is continuous in (z, w), $z \neq w$). Then from (2.5)

$$0 = \lim_{n \to \infty} \left(\inf_{z \in K} \frac{1}{k(n)} \sum_{\nu} G(z, z_{\nu,n}) \right)$$
$$\geqslant \limsup_{n \to \infty} (\delta \cdot \nu_n(S)). \tag{2.7}$$

Thus

$$\lim_{n \to \infty} \nu_n(S) = 0, \tag{2.8}$$

which implies that there are only $o(k(n))$ zeros in S.

Since (ν_n) is a sequence of unit measures on the Borel sets of $\bar{\mathbb{C}}$, every subsequence of (ν_n) possesses a weak limit point. Let ν be the weak limit of the subsequence $(\nu_{l(n)})$. It remains to show that $\nu = \mu_E$.

Let $z_0 \in \mathbb{C} \setminus E$ and $U = S \cup V$ the union of a circle S around z_0 and a neighborhood V of infinity, such that $U \subseteq \bar{\mathbb{C}} \setminus E$ and $\bar{\mathbb{C}} \setminus (E \cup U)$ is connected. Then (1.9) holds with K replaced by S. Decompose $p_n = \tilde{p}_n \cdot q_n$, such that

$$q_n(z) = a_n \cdot \prod_{\nu = 1}^{m(n)} (z - z_{\nu,n}), \tag{2.9}$$

where $z_{\nu,n}$, $\nu = 1, ..., m(n)$, are the zeros of p_n in U and a_n is the highest coefficient of p_n.

Fix $\varepsilon > 0$. Take $R_0 > 0$, such that $|z| \geqslant R_0$ implies $z \in V$. By the maximum principle for subharmonic functions, (1.7) and (2.8),

$$\begin{aligned} \varepsilon &> h_n(z) \\ &= \frac{1}{k(n)} \log |\tilde{p}_n(z)| - G(z) + \frac{1}{k(n)} \log |q_n(z)| \\ &\geqslant \log \operatorname{cap} E - \varepsilon + \frac{1}{k(n)} \log |q_n(z)|, \end{aligned}$$

for $|z| = R_\varepsilon > R_0$ and $n \geqslant N_\varepsilon$, where R_ε and N_ε are chosen large enough. By the maximum principle applied to $q_n(z)$,

$$\limsup_{n \to \infty} \left(\sup_{z \in S} \frac{1}{k(n)} \log |q_n(z)| \right) \leqslant -\log \operatorname{cap} E. \tag{2.10}$$

Let $\tilde{\nu}_n$ denote the zero-measure associated with $\tilde{p}_n$. Then, by (2.8), ν is a weak limit point of $\tilde{\nu}_{l(n)}$. Thus, for all $z \in S^\circ$,

$$\lim_{n \to \infty} U^{\tilde{\nu}_{l(n)}}(z) = -\lim_{n \to \infty} \int_{\mathbb{C} \setminus U} \log |z-\zeta| \, d\tilde{\nu}_{l(n)}(\zeta)$$

$$= -\int \log |z-\zeta| \, d\nu(\zeta)$$

$$= U^\nu(z). \tag{2.11}$$

Let $\tilde{S} \subseteq S^\circ$ be a compact neighborhood of z_0, such that $\partial\tilde{S}$ is regular. Then the convergence in (2.11) is uniform in $\tilde{S}$. Furthermore, (1.9) holds for $\tilde{S}$ too. Let $\varepsilon > 0$. Then there is an $N \in \mathbb{N}$, such that for $n \geqslant N$

$$\frac{1}{k(n)-m(n)} \log |q_n(z)| \leqslant -\log \operatorname{cap} E + \varepsilon \qquad \text{for all } z \in S \tag{2.12}$$

$$\sup_{z \in \tilde{S}} \left(\frac{1}{k(n)-m(n)} \log |p_n(z)| - G(z) \right) \geqslant -\varepsilon, \tag{2.13}$$

where we used (2.10), (1.9), and $m(n)/k(n) \to 0$. Thus, for $n \geqslant N$,

$$\inf_{z \in \tilde{S}} U^{\tilde{\nu}_n}(z) = \inf_{z \in \tilde{S}} -\frac{1}{k(n)-m(n)} \log |\tilde{p}_n(z)|$$

$$\leqslant \sup_{z \in \tilde{S}} (-\log \operatorname{cap} E - G(z) + 2\varepsilon)$$

$$= \sup_{z \in \tilde{S}} U^{\mu_E}(z) + 2\varepsilon. \tag{2.14}$$

By (2.11) and the uniform convergence in $\tilde{S}$, we can deduce

$$U^\nu(z) \leqslant U^{\mu_E}(z) \qquad \text{for} \quad z \in \mathbb{C} \setminus E, \tag{2.15}$$

since $\tilde{S}$ and $\varepsilon > 0$ was arbitrary. Since $(U^\nu - U^{\mu_E})(\infty) = 0$, we get

$$U^\nu(z) = U^{\mu_E}(z) \qquad \text{for all} \quad z \in \mathbb{C} \setminus E. \tag{2.16}$$

Now we reproduce an argument in [1]. Denote by

$$I[\mu] = \int U^\mu(z) \, d\mu \tag{2.17}$$

the energy of the measure μ. Then μ_E is the unique measure, supported on ∂E, which minimizes $I[\mu]$ (compare [6]). Since U^{ν} is lower semi-continuous on ∂E, we get

$$U^{\nu}(z) \leqslant U^{\mu_E}(z) \qquad \text{for} \quad z \in \partial E. \tag{2.18}$$

Since ν is supported on ∂E,

$$I[\nu] = \int U^{\nu}\, d\nu \leqslant \int U^{\mu_E}\, d\nu = I[\mu_E]. \tag{2.19}$$

Thus $\nu = \mu_E$ and the proof of Theorem 1 is complete.

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References

1. H. P. Blatt, E. B. Saff, and M. Simkani, Jentzsch-Szegö type theorems for the zeros of best approximants, *J. London Math. Soc. (2)* **38** (1988), 307–316.
2. P. Erdős and P. Turán, On the distribution of roots of polynomials, *Ann. of Math.* **51** (1950), 105–119.
3. R. Grothmann and E. B. Saff, On the behavior of zeros and poles of best uniform polynomial and rational approximation, *in* "Nonlinear Methods and Rational Approximation" (A. Cuyt, Ed.), pp. 57–75, 1988.
4. R. Jentzsch, "Untersuchungen zur Theorie Analytischer Funktionen," Inaugural-Dissertation, Berlin, 1914.
5. G. Szegö, Über die Nullstellen von Polynomen, die einem Kreis gleichmässig konvergieren, *Sitzungsber. Ber. Math. Ges.* **21** (1922), 59–64.
6. M. Tsuji, "Potential Theory in Modern Function Theory," Chelsea, New York, 1958.
7. M. Simkani, Accumulation of zeros of approximating and interpolating polynomials, *in* "Approximation Theory" (C. K. Chui, L. L. Schumaker, and J. D. Ward, Eds.), Vol. 5, 1986.
8. J. L. Walsh, "Interpolation and approximation," American Math. Soc. Colloquium Publications, New York, 1935.
9. J. L. Walsh, The analogue for maximally convergent polynomials of Jentsch's theorem, *Duke Math. J.* **26** (1959), 605–616.